PHYSICAL REVIEW E **67**, 030301 (2003)

Oslo rice pile model is a quenched Edwards-Wilkinson equation

Gunnar Pruessner*
Department of Mathematics, Imperial College, 180 Queen's Gate, London SW7 2BZ, United Kingdom
(Received 14 September 2002; published 19 March 2003)

The Oslo rice pile model is a sandpile-like paradigmatic model of self-organized criticality (SOC). In this paper it is shown that the Oslo model is in fact *exactly* a discrete realization of the much studied quenched Edwards-Wilkinson equation (qEW) [Nattermann *et al.*, J. Phys. II France **2**, 1483 (1992)]. This is possible by choosing the correct dynamical variable and identifying its equation of motion. It establishes for the first time an exact link between SOC models and the field of interface growth with quenched disorder. This connection is obviously very encouraging as it suggests that established theoretical techniques can be brought to bear with full strength on some of the hitherto elusive problems of SOC.

DOI: 10.1103/PhysRevE.67.030301 PACS number(s): 45.70.Ht, 05.65.+b, 64.60.Ht, 68.35.Fx

The Oslo rice pile model (Oslo model hereafter) was originally intended to model the relaxation processes in real rice piles [1]. Meanwhile, it has been subject to many investigations and publications in its own right. The model as defined below supposedly develops into a scale-free state without the explicit tuning of external parameters, and is therefore regarded as an example of self-organized criticality (SOC) [2]. In fact, contrary to many other ''standard'' models of SOC [3–6], it shows a reliable and consistent (simple) scaling behavior and is robust against certain changes in the details of the dynamics [7–9]. The most prominent observable in the model, the avalanche size s, is governed by a probability distribution $\mathcal{P}(s)$ which obeys simple scaling,

$$\mathcal{P}(s)=s^{-\tau}\mathcal{G}(s/s_0) \quad \text{and} \quad s_0=L^D, \tag{1}$$

where L denotes the system size and τ and D are critical exponents, consistently reported to be $\tau=1.55(10)$ and $D=2.25(10)$ [7–12]. These two exponents are related by $D(2-\tau)=1$ [10,11], which can be proven easily given that the first moment of $\mathcal{P}(s)$, $\langle s\rangle$, scales like L.

In the following the model is defined, the relevant dynamical variable extracted and its equation of motion derived, which turns out to be a discretized quenched Edwards-Wilkinson (qEW) equation. By analyzing the essential characteristics of the model on the lattice, such as uniqueness of the solution and symmetries, it is then possible to construct the continuum theory, which can subsequently be examined using standard methods.

The model [10] is defined on a one-dimensional grid of size L, where each site $i=1\cdots L$ has slope z_i and critical slope $z_i^c\in\{1,2\}$. Starting from an initial configuration with $z_i=0$ and z_i^c random everywhere, the model evolves according to the following update rules: (1) (Driving) Increase z_1 by one. (2) (Toppling) If there is an i with $z_i>z_i^c$ decrease z_i by 2 and increase its nearest neighbors by one, $z_{i\pm1}\to z_{i\pm1}+1$, provided that $1\leqslant i\pm1\leqslant L$. A new z_i^c is chosen at random, 1 with probability p and 2 with probability $q\equiv1-p$. (3) Repeat the second step until $z_i\leqslant z_i^c$ everywhere. Then proceed with the first step. The order of updates is irrelevant in this model and the original definition does not fix it explicitly. Therefore the microscopic (fast) timescale is *a priori* undefined.

The avalanche size s is defined as the number of charges, i.e., apart from boundary effects, twice the number of times the second rule is applied between two consecutive applications of the first rule. For convenience the model is dissipative on both boundaries, where one of the two ''units'' lost by the boundary site during toppling leaves the system.

A few years ago Paczuski and Boettcher translated the Oslo model into the language of interfaces in random media [11]. However, the evolution of the dynamical variable $H(x,t)$, which is the total number of topplings of site x, was given by $\partial_t H=\theta(\partial_x^2 H-\eta(x,H))$, where ∂_t is defined in discrete time, i.e., $\partial_t H\equiv H(x,t+1)-H(x,t)$ and ∂_x^2 is the lattice Laplacian, so that x is actually an index. The last term $\eta(x,H)$ represents a quenched noise. The Heaviside θ function makes this equation of motion highly nonlinear and analytically almost intractable [13]. Paczuski and Boettcher have already conjectured that the Oslo model is in the same universality class as qEW [14]. More recently, Alava has suggested that certain other sandpile models are described by qEW [15]. It is, however, important to realize that no rigorous and exact link has so far been established between SOC models and the qEW equation.

The crucial step to make this correspondence exact is to identify the proper dynamical variable. It is found in the form of the number of times a site has been charged (i.e., received a unit from a neighbor during a toppling or by external drive, see below) $h(x,t)$, where x and t are discrete for the time being. There is a simple functional relation between $h(x,t)$ and $H(x,t)$, which can be obtained as follows: Each site can be in one of three stable configurations, $z_i\in0,1,2$. When a site receives a unit from a neighbor, it changes state as shown in Fig. 1. Charging a site in state 0 necessarily leads to state 1 without toppling and the specific value of z_i^c is completely irrelevant at this stage. Similar for state 2: If a site receives a charge in this state, its z_i^c must be 2 and it must topple. The only point where the value of z_i^c actually matters, is in state 1, therefore it can be effectively chosen at random when necessary, so that the site topples with probability p (according to the probability of having $z_i^c=1$) or

*Electronic address: gunnar.pruessner@physics.org

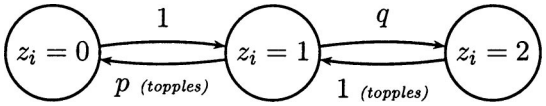

FIG. 1. Each site can be in one of three states and changes stepwise between them, whenever it receives a charge. The labels indicate the probability of the move and whether it entails a toppling.

increases to 2 with probability q (see Fig. 1). It is immediately clear that any even number of charges, say $m=2n$, starting from $z_i=1$ leads to state 1 again with n topplings. An odd number of charges, say $m=2n+1$, leads either to n topplings and state 2 or $n+1$ topplings and state 0. This is illustrated in Fig. 1: The m charges lead to m steps along the arrows. Whenever one moves left, the site topples.

In order to write a functional relation between $h(x,t)$ and $H(x,t)$, the randomness in the decision of moving to the left or to the right from state 1 must be quenched in $h(x,t)$, i.e., it is not allowed to change unless $h(x,t)$ changes. This can be summarized as

$$H(x,t+1)=\frac{1}{2}[h(x,t)+\eta(x,h(x,t))], \tag{2}$$

where η is 0 whenever $h(x,t)$ is even, corresponding to state 1. If $h(x,t)$ is odd, η is either 1 (with probability p, state $z_i=0$) or $-1(z_i=2)$. Every sequence of $\eta(x,h)$ values maps uniquely to a sequence of z_i^c and vice versa. The equation above can easily be transformed to comply to any initial configuration, especially to $z_i(t=0)\equiv 0$. Essentially, it is (2), which makes the exact identification of the Oslo model and qEW possible.

The final equation is derived by noting that obviously $h(x,t)=H(x-1,t)+H(x+1,t)$ with appropriately chosen boundary conditions (BC's) (see below), so that using the short-hand notation $h^{\pm}=h(x\pm 1,t)$ and $\eta^{\pm}=\eta(x\pm 1,h^{\pm})$ the equation of motion is

$$h(x,t+1)-h(x,t)=\frac{1}{2}(h^{-}-2h(x,t)+h^{+}+\eta^{+}+\eta^{-}), \tag{3}$$

which is the *exact* representation of the Oslo model as defined above, captured in a single equation. Its differential form is accordingly

$$\partial_t h(x,t)=\frac{1}{2}\partial_x^2 h(x,t)+\left(1+\frac{1}{2}\frac{d^2}{dx^2}\right)\eta(x,h(x,t)). \tag{4}$$

The right hand BC is $h(x=L+1,t)\equiv 0$ [and $h(x=L,t)\equiv 0$ in the continuum], while the left hand BC provides the driving via $h(x=0,t)=2E(t)$, $E(t)$ being the total number of initial seeds (step 1 above) at time t. These seeds arrive at site $x=1$ via the Laplacian. In the continuum, the simplest drive is $E(t)=vt$ with v a driving velocity and t the microscopic time. Together with the BC's, Eq. (4) or the generalized form

$$\partial_t h(x,t)=\kappa\partial_x^2 h(x,t)+g\left(1+\lambda\frac{d^2}{dx^2}\right)\eta(x,h(x,t)), \tag{5}$$

where the correlator of η is now normalized, i.e., $\int dx\int dh\langle\eta\eta\rangle=1$, describes the movement of an elastic band over a rough surface [16] pulled by a transverse force acting at one end point only. Below it is shown that the λ term disappears in the continuum, establishing the *first rigorous identification* of the Oslo model and the qEW equation. The same equation with different properties of the noise term and/or different BC's applies to other models, such as the Bak-Tang-Wiesenfeld (BTW) model [17], fixed energy sandpiles (for example [18]) or the tilted sandpile [19]. Having identified the relevant dynamical variable h, the effect of modifications of the dynamical rules of the Oslo model, such as [7–9], can be understood.

The equation above exemplifies a general ''trick'' [29] to get rid of θ functions in equations of motion—they often appear in descriptions of sandpile-like systems (for example [13]): One simply replaces $\theta(h-h_c)$ by $h+\eta(h)$ with an appropriately chosen sawtooth like η. This does not necessarily simplify the problem, unless there is already a quenched noise present in the system. In this case the θ turns into a correlation in η. This is highly remarkable from the point of view of SOC, because the presence of ''thresholds'' is usually expected to be a crucial ingredient of SOC [2,17,20]. Moreover, the correlations in η, which are of fundamental significance in interface models [14,21] and have been neglected in former mappings, now arise naturally from the dynamical description of the model.

In order to construct the proper continuum theory, it is worthwhile to consider the formal solution of Eq. (5). It will turn out later that $E(t)=vt$ is sufficiently general, so that it makes sense to define $v(x)\equiv v(L-x)/L$ and

$$h(x,t)=2v(x)t+P_3(x)+z(x,t) \tag{6}$$

in order to homogenize the BC's. $P_3(x)$ is a third-order polynomial only present to cancel the first term in the differential equation, i.e., $\kappa\partial_x^2 P_3=2v(x)$, with roots at $x=0$ and $x=L$. Therefore $\partial_t z=\kappa\partial_x^2 z+g\,\eta_\lambda(x,h(x,t))$ with homogenous BC's. The term $\eta_\lambda(x,h(x,t))\equiv(1+\lambda(d^2/dx^2))\eta(x,h(x,t))$ is actually a functional of h. The initial condition of $z(x,t)$ is not $z(x,t=0)\equiv 0$ as for h, because of the data shift above. But due to the homogenous BC's any initial condition decays, so that the initial sources, accounting for $z(x,t=0)=-P_3(x)$, can be ignored. Then the formal solution is $z(x,t)=\sum_{n=1}^{\infty}z_n(t)\sin(k_n x)$ with

$$z_n(t)=\frac{2g}{L}\int_0^t dt'\int_0^L dx'\,\eta_\lambda(x',2v(x)t+z(x',t))\times\sin(k_n x)\exp(-k_n^2\kappa(t-t')) \tag{7}$$

and $k_n=\pi n/L$.

According to Eq. (6), the tilt of $h(x,t)$ in x increases in time. Assuming stationarity of the relevant statistical properties (especially avalanches as defined below), this requires the solution to be invariant under tilt, which is also known as

Galilean invariance [22]: $h'=h+\alpha x$ must produce the same statistics as h, which entails $\eta(x,a+\alpha x)$ to be equally likely as $\eta(x,a)$, so that $\langle \eta(x,a+\alpha x)\,\eta(x',a'+\alpha x')\rangle = \langle \eta(x,a)\,\eta(x',a')\rangle$. But assuming the standard form [14] $\langle \eta(x,a)\,\eta(x',a')\rangle = \Delta_{\parallel}(x-x')\Delta_{\perp}(a-a')$, the correlator obeys for any $x-x'$ where $\Delta_{\parallel}(x-x')$ is finite, $\Delta_{\perp}(a-a') = \Delta_{\perp}(a-a'+\alpha(x-x'))$. This holds for any α, so if $\Delta_{\parallel}(x-x')$ was finite for any $x-x'\neq 0$, $\Delta_{\perp}$ would be bound to be a constant. This is impossible, because $\Delta_{\perp}$ must be nonvanishing somewhere and normalizable, so that $\Delta_{\parallel}(x-x')$ must vanish for any finite $x-x'$, i.e., it *must* be a δ function.

Next it can be shown that the Oslo model obeys Middleton's no-passing [23]. For $\lambda\neq 0$ this will lead to a constraint on the noise which is incompatible with the δ correlation of $\Delta_{\parallel}$ in the continuum, so that λ must vanish in the continuum. Defining a partial ordering $\geqslant$ for two configurations $h_1(t_1,x)$ and $h_2(t_2,x)$ of the interfaces as $h_1(t_1,x) \geqslant h_2(t_2,x) \Leftrightarrow \forall_{x\in[0,L]} h_1(t_1,x) \geqslant h_2(t_2,x)$, one has to show that this order is preserved under the dynamics [24]. With the "external field" being the BC's $E_1(t)$ and $E_2(t)$, one shows that if $h_1(t_0,x)\geqslant h_2(t_0,x)$ for a given t_0 [which entails $E_1(t_0)\geqslant E_2(t_0)$] the interfaces can never "overtake" each other at $t\geqslant t_0$. By assuming the opposite, one only needs to prove that where the two interfaces "touch" for the first time, x_0, the velocity of h_1 is higher or equal to the velocity of h_2. For the model on the lattice (3), this is equivalent to

$$h_1^+ + \eta_1^+ + h_1^- + \eta_1^- \geqslant h_2^+ + \eta_2^+ + h_2^- + \eta_2^- \tag{8}$$

using the same notation as in Eq. (3). In the original discrete model, condition (8) follows immediately from $\eta(x,h)+h$ being a monotonically increasing function in h for any x. For the continuum equation (5) the corresponding calculation gives

$$\lambda g \partial_h \eta(x,h) \geqslant -\kappa \tag{9}$$

assuming that $(d^2/dx^2)\,\eta = \partial_x^2\eta + \partial_x h\partial_x\partial_h\eta + \partial_x h\partial_h\partial_x\eta + \partial_x^2 h\partial_h\eta + (\partial_x h)^2\partial_h^2\eta$ and that the interface is smooth in x_0 such that $\partial_x h_1(x_0,t) = \partial_x h_2(x_0,t)$ and $\partial_x^2 h_1(x_0,t) > \partial_x^2 h_2(x_0,t)$. For a noise with divergent width, $\Delta_{\parallel}(x) = \delta(x)$, Eq. (9) cannot hold for any $\lambda\neq 0$, i.e., a nonvanishing λ destroys no-passing. However, no-passing must be regarded as a crucial feature, as it ensures the asymptotic uniqueness of the configuration and is reminiscent of the irrelevance of the order of updates in the original model, so that $\lambda=0$ is a necessary condition for the equivalence of the continuum and discrete model.

This is physically justified: Assuming a smooth η, in the continuum approximation of Eq. (3), λ becomes proportional to the square of the lattice spacing and therefore vanishes in the continuum limit.

Keeping the λ term nevertheless, a naïve scaling analysis shows that it is irrelevant. Moreover, its Fourier transform in Eq. (7) produces only a term $-g\lambda k_n^2$, because of the total derivative in η_λ. This can be absorbed into the bare propagator of a perturbative expansion in the style of [14,21] in the form $2g(1-\lambda k_n^2)/L(\kappa k_n^2 + i\omega)$, leading possibly to an ultraviolet divergence. Apart from that, the terms obtained for an renormalization group treatment are structurally the same as in [21] as calculations show (details to be published later). The only differences are due to the peculiar way of driving the interface [i.e., the term $2v(x)$, which is a mean velocity in (6), but also drives the model by moving the quenched noise in (7)] and the nonconservative nature of the interface (which makes sense only for a finite system) leading to the homogenous BC's and therefore to the $\sin(k_n x)$ rather than $\exp(2ik_n x)$ terms. In turn, the standard qEW problem [14] corresponds to an Oslo model with periodic BC's and continuous, uniform drive.

Expanding η in powers of z_n, the first two terms of $z_n(\omega)$ (the Fourier transform of (7) in t) are

$$z_n(\omega) = \frac{2g(1-\lambda k_n^2)}{L(\kappa k_n^2 + i\omega)}\left[\int_0^L dx'\,\hat{\eta}\left(x', \frac{\omega}{2v(x')}\right)\frac{\sin(k_n x')}{2v(x')} + \int_0^L dx' \int_{-\infty}^{\infty} dq \sum_{m=1}^{\infty} \hat{\eta}(x',q)\,\frac{iq\sin(k_m x')}{\sqrt{2\pi}} \times z_m(\omega - 2v(x')q)\sin(k_n x')\right],$$

where $\hat{\eta}(x,q)$ is the Fourier transform of $\eta(x,h)$ in h.

The definition of the avalanche size s in the continuum is the area between the interface configurations at two times t_1 and t_2, $s=\int_0^L dx[h(x,t_2)-h(x,t_1)]$, so that $\langle s\rangle = v\Delta t L$ with $\Delta t \equiv t_2 - t_1$, because $\langle z(x,t)\rangle$ is expected to be asymptotically independent of t, as a nonvanishing $\lim_{t\to\infty}\partial\langle z(x,t)\rangle$ with homogenous BC's would require support for a divergent curvature of the interface. Choosing $\Delta h \equiv \Delta t v$ constant for different system sizes L then preserves the property $\langle s\rangle \propto L$.

Due to the asymptotic uniqueness of the solution the system can either be driven in jumps of Δh separated by suffi-

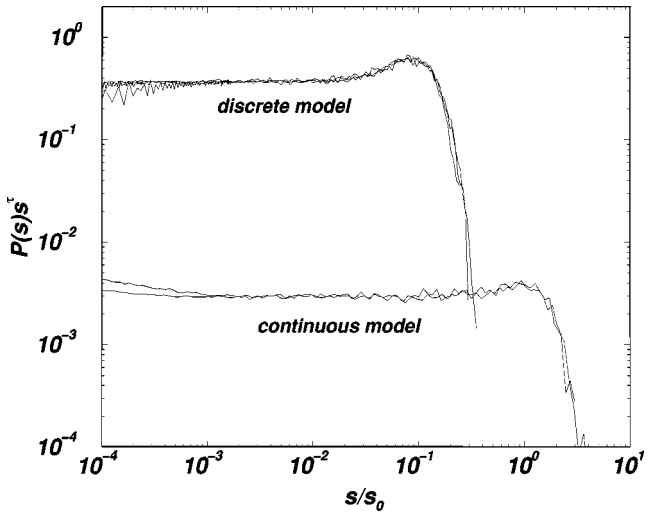

FIG. 2. Comparison of a data collapse according to (1) for system sizes between $L=128$ and $L=512$ and the continuous and the discrete realization of the model. The same value of $\tau=1.55$ collapses all curves within each model onto its scaling function. Due to the omission of the nonuniversal constants Eq. (1) the two resulting curves are shifted relative to each other.

ciently long times, or driven very slowly taking ''snapshots'' of the configuration in order to calculate s.

The model possesses two characteristic timescales: One is the diffusive timescale $t_0 \equiv L^2/\kappa$, the other one is the nontrivial scale due to noise and drive, $t_g \equiv g^2/(v^3 L)$. One has to maintain a sufficiently large Δt to prevent distinct avalanches from merging, otherwise the central limit theorem would turn $\mathcal{P}(s)$ into a Gaussian. The SOC limit is usually identified with $v \to 0$, which makes sense only in the presence of an intrinsic scale for v. The only combination of parameters (κ, g and L, but $\lambda = 0$) which provides a ''natural velocity'' is $v_g \equiv (g^2\kappa)^{1/3}/L$. The SOC condition $v \to 0$ is therefore already met by $v \ll v_g \propto L^{-1}$, which is however, not sufficient. According to Ref. [11] $\Delta t \gg L^z$ with $z \approx 1.42$, so that $\Delta h = \text{const.}$ entails $v \ll L^{-z}$, which therefore seems to be the correct condition for SOC, even though the microscopic timestep in [11] is defined as a parallel update, which is not *exactly* (3).

Preliminary numerical studies indeed suggest that (5) with $\lambda = 0$ is a valid continuous description of the Oslo model: Fig. 2 compares a scaling collapse for different system sizes of the continuous model (with $\lambda = 0$) and the original, discrete one. The best collapse is obtained by $\tau = 1.55$ for both models. The scaling law $D = 1 + \chi$ [11] remains applicable as long as the two configurations at t_1 and t_2 are correlated. It is in perfect agreement with numerical results [25,26] for the qEW model [30].

In conclusion, the Oslo model has been reduced to a quenched Edwards-Wilkinson equation. In the continuum limit the qEW becomes the *exact* equation of motion for the Oslo model. This not only makes it possible to approach the exponents of an SOC-model analytically, but also gives insight into the nature of avalanche like behavior and the relation between SOC and other theories of critical phenomena. It provides the perfect test bed for analytical methods proposed for SOC.

The established relationship is presently being pursued in order to develop a direct approach to the critical exponent τ, clear up the rôle of the noise and clarify the relation between noise and drive. The framework used here is also promising for other models, such as the BTW model [17], various other sandpile models [18,19] and the Zhang model [27].

The author wishes to thank Nicholas R. Moloney for proofreading, Henrik J. Jensen for suggesting the problem and for very helpful discussions, as well as Alvin Chua and Kim Christensen for presenting and discussing their work [28] prior to publication. The author gratefully acknowledges the support of the EPSRC and the NSF during the 2001 Boulder School for Condensed Matter and Material Physics.

[1] V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang, and P. Meakin, Nature (London) **379**, 49 (1996).
[2] H. J. Jensen, *Self-Organized Criticality* (Cambridge University Press, New York, 1998).
[3] A. S. Datta, K. Christensen, and H. J. Jensen, Europhys. Lett. **50**, 162 (2000).
[4] P. L. Dorn, D. S. Hughes, and K. Christensen (2001), preprint available from http://www.cmth.ph.ic.ac.uk/kim/papers/preprints/preprint_btw.pdf.
[5] S. Lise and M. Paczuski, Phys. Rev. E **64**, 046111 (2001).
[6] G. Pruessner and H. J. Jensen, Phys. Rev. E **65**, 056707 (2002); cond-mat/0201306.
[7] S. Zhang, Phys. Lett. A **233**, 317 (1997).
[8] M. Bengrine, A. Benyoussef, A. E. Kenz, M. Loulidi, and F. Mhirech, Eur. Phys. J. B **12**, 129 (1999).
[9] M. Bengrine, A. Benyoussef, F. Mhirech, and S. D. Zhang, Physica A **272**, 1 (1999).
[10] K. Christensen, A. Corral, V. Frette, J. Feder, and T. Jøssang, Phys. Rev. Lett. **77**, 107 (1996).
[11] M. Paczuski and S. Boettcher, Phys. Rev. Lett. **77**, 111 (1996).
[12] A. Corral and M. Paczuski, Phys. Rev. Lett. **83**, 572 (1999).
[13] A. Díaz-Guilera, Phys. Rev. A **45**, 8551 (1992).
[14] T. Nattermann, S. Stepanow, L.-H. Tang, and H. Leschhorn, J. Phys. II **2**, 1483 (1992).
[15] M. Alava, J. Phys.: Condens. Matter **14**, 2352 (2002); cond-mat/0204226.
[16] M. Dong, M. C. Marchetti, A. A. Middleton, and V. Vinokur, Phys. Rev. Lett. **70**, 662 (1993).
[17] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
[18] A. Vespignani, R. Dickman, M. A. Muñoz, and S. Zapperi, Phys. Rev. E **62**, 4564 (2000); cond-mat/0003285.
[19] A. Malthe-Sørenssen, Phys. Rev. E **59**, 4169 (1999).
[20] R. Calfiero, V. Loreto, L. Pietronero, A. Vespignani, and S. Zapperi, Europhys. Lett. **29**, 111 (1995).
[21] H. Leschhorn, T. Nattermann, S. Stepanow, and L.-H. Tang, Ann. Phys. (Leipzig) **6**, 1 (1997); cond-mat/9603114.
[22] P. Meakin, *Fractals, Scaling and Growth Far from Equilibrium* (Cambridge University Press, Cambridge, 1998).
[23] A. A. Middleton, Phys. Rev. Lett. **68**, 670 (1992).
[24] J. P. Sethna, K. Dahmen, S. Kartha, J. A. Krumhansl, B. W. Roberts, and J. D. Shore, Phys. Rev. Lett. **70**, 3347 (1993).
[25] H. Leschhorn, Physica A **195**, 324 (1993).
[26] P. Le Doussal, K. J. Wiese, and P. Chauve, Phys. Rev. B **66**, 174201 (2002); cond-mat/0205108.
[27] Y.-C. Zhang, Phys. Rev. Lett. **63**, 470 (1989).
[28] A. Chua and K. Christensen, cond-mat/0203260.
[29] The substitution was already used earlier (see [15] and references therein).
[30] For $\chi = 5/4$ one has $D = 9/4$ and $\tau = 14/9$.